Strategies for Mathematical Reasoning

Mathematical Reasoning

Mathematical reasoning refers to the logical thinking skills that individuals develop while learning mathematics. Developing mathematical reasoning skills enables students to gain skills in the classroom that they can then transfer to problem solving in all areas of their lives. Through mathematical reasoning, students learn how to evaluate a problem, design a plan to solve the problem, execute that plan, and then evaluate the results and make adjustments as needed. These same reasoning skills can then be applied to real-life problems on the job, at home, within the community, or in higher education and training programs.

Overview Mathematical Reasoning Test

The GED® Mathematical Reasoning Test focuses on two major content areas:

- quantitative problem solving – approximately 45%
- algebraic problem solving – approximately 55%

The area of descriptive statistics and basic inference are embedded primarily in the Science and Social Studies Tests. However, assessment targets in the area of data and statistics are also included in Mathematical Reasoning.

In addition to the content-based assessment targets, the Mathematical Reasoning Test also focuses on mathematical practices. Mathematical practices describe the types of behaviors in mathematics that are essential to mastering mathematical content. Modeling is one of the

Strategies for Mathematical Reasoning

"The only way to learn mathematics is to do mathematics."

Paul Halmos

most important behaviors, which emphasizes applying mathematics to real-life situations as well as to academic problems in other fields of study.

The Mathematical Reasoning Test features: multiple choice items, a variety of technology-enhanced item types, and drop-down items. A virtual scientific calculator, the TI-30XS MultiView™, is embedded in the computer-based delivery platform. Candidates will be able to use the calculator for all but the first five items on the test.

Mathematical Content Standards

Each assessment target on the Mathematical Reasoning Test corresponds with one or more domains from the Common Core State standards (CCSS) of Mathematic and a mathematical practice.

At the high school level, the Common Core State Standards for Mathematics are divided by:

- Content Standards (6)
- Practice Standards (8)

Mathematical Content Standards

- Number and Quantity
- Modeling
- Algebra
- Functions
- Geometry
- Statistic and Probability

Mathematical Practice Standards

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

It's important to familiarize yourself with the various educational standards that are the foundation of the Common Core State Standards and the 2014 GED® test.

Resources

For additional information on the Mathematical Reasoning Assessment Targets, access the Assessment Guide for Educators, Chapter 2:

- GED Testing Service® website
  http://www.gedtestingservice.com/educators/assessment-guide-for-educators

For additional information on the Florida GED® Curriculum Frameworks access:

- Florida's GED® Curriculum Frameworks
  http://www.fldoe.org/workforce/dwdframe/ad_frame.asp

For additional information on Common Core State Standards, access the Mathematics standards at:

- Common Core State Standards
  http://www.corestandards.org/Math

Standards-Driven Curriculum

Standards provide adult educators with a starting point – a method for increasing awareness and understanding of the skills and knowledge that adult learners must have to be successful as they pursue postsecondary education and training and employment. Although standards do not tell individual schools or instructors in the classroom what they must teach, standards do provide a consistent, clear understanding of what students are expected to learn, so teachers know what they need to do to help their students.

The *College and Career Readiness (CCR) Mathematical Standards* identify three key shifts: focus, coherence, and rigor.

Shift 1 – Focus: Focusing strongly where the standards focus

This shift requires that classroom instruction focuses on:

- key ideas, understandings and skills identified by the standards
- deep learning of concepts
- learning fewer concepts with more depth, rather than learning the same skills repeatedly without mastery

Shift 2 – Coherence: Designing learning around coherent progressions level to level

This shift focuses on the need for classrooms to design learning around coherent progressions where similar standards exists between the different levels, but the focus and rigor change as students progress.

Shift 3 – Rigor: Pursuing conceptual understanding, procedural skill and fluency, and application

This shift focuses on the need for understanding and applying concepts. Rigor is more than just procedural skill and fluency. Rather, rigor is the understanding and application of conceptual knowledge that leads to students being able to reason abstractly.

CCR Mathematical Standards

Mathematical Practices

Mathematical practices refer to the way mathematics is made and used in real-world situations. Therefore, mathematical content needs to be taught through the practices - the way in which math is used in our daily lives. This provides real connections for learning and applying what has been learned.

There are eight mathematical practices – grouped as:

- *habits of mind*
    - make sense of problems and persevere in solving them
    - attend to precision
- *reasoning and explaining*
    - reason abstractly and quantitatively
    - construct viable arguments and critique the reasoning of others
- *modeling and using tools*
    - model with mathematics
    - use appropriate tools strategically
- *seeing structure and generalizing*
    - look for and make use of structure
    - look for and express regularity in repeated reasoning

Using standards to drive both curriculum and instructional strategies is important when developing a GED® preparatory program. Note that these are the same mathematical practices as those of the Common Core State Standards.

Resources

For additional information on the Florida GED® Curriculum Frameworks access:

- Florida's GED® Curriculum Frameworks
  http://www.fldoe.org/workforce/dwdframe/ad_frame.asp

Strategies for the Classroom

There is a proverb that states, "What I hear, I forget; what I see, I remember; what I do, I understand." This proverb is a fundamental principle of active learning. Extensive research has shown that students learn more rapidly, retain knowledge longer, and develop superior critical thinking skills when they are actively involved in the learning process. This is especially true with mathematics.

The premise that "students learn math by doing math, not by listening to someone talk about doing math" (Twigg 2005)[16] provides the approach for teaching mathematical reasoning skills. Successful instruction in quantitative reasoning and algebraic problem solving requires a progressive approach where instruction connects content to real-life situations with an emphasis on a deeper understanding of concepts, rather than facts, and then applying those concepts to real-world situations.

Teaching Mathematics: An Introduction

- Students are often fearful regarding mathematics. Research states that it is both the most disliked and the most loved subject area. It is important to address and evaluate student attitudes and beliefs regarding both learning math and using math. Prior to any true learning taking place, discuss with students how methods of teaching mathematics may have caused them to develop a negative attitude.

- Determine what students already know about a topic before instruction. Use an informal discussion of what students already know about a topic prior to teaching. Formal assessment instruments do not always provide an accurate picture of a student's real life knowledge or thinking processes. For example, if discussing positive and negative integers, discuss a bank account and the concept of being "overdrawn" or in the negative category.

- Develop understanding by providing opportunities to explore mathematical ideas with concrete or visual representations and hands-on activities. Students will learn more

[16] Twigg, Carol A. (2005). Math Lectures: An Oxymoron? The National Center for Academic Transformation

effectively if they can visualize concretely an abstract concept (If you can "see" it, you can solve it).

- Use manipulatives such as Cuisenaire rods, fraction circles, geoboards, algebra tiles, or everyday objects such as coins, toothpicks, etc. to help students explain how mathematical rules and concepts work. Start with concrete objects to move to abstract ideas.

- Encourage the development and practice of estimation skills and mental math. Although not directly assessed on the Mathematical Reasoning Test, these skills are useful both in determining the reasonableness of an answer, as well as during everyday life when one does not always use "exact" math.

- Integrate problem-solving abilities. Word problems or real-life problems must be a significant part of instructional time. Have students write their own word problems to reinforce the connection between mathematical content and real-world application.

- Encourage use of multiple solution strategies. Teach students how to solve problems in different ways. Ensure that more than one strategy is used to solve a problem so that students are comfortable in integrating different problem-solving solutions. Categories of problem solving skills include:
    - Drawing a picture or diagram
    - Marking a chart or graph
    - Dividing a problem into smaller parts
    - Looking for patterns
    - Using a formula or written equation
    - Computing or simplifying
    - Using the process of elimination
    - Working backwards

- Help students understand the process required in problem solving. Use a strategy that is adapted from the Polya Problem Solving Strategy:
    - Understand the problem
    - Devise a plan
    - Carry out the plan
    - Look back[17]

- Provide opportunities for group work. Develop a project where a group effort is appropriate. An example would be to organize an activity where the development of a plan, schedule, budget, needed business materials, and a report would be required. As

[17] Polya, G. (1954). *How to Solve It*. (2nd ed.). Princeton, NJ: Princeton University Press
Polya. G. (1980). *On Solving Mathematical Problems in High School*. In S. Krulik (Ed.), Problem Solving in School Mathematics. Reston, VA: NCTM

with all group activities, clear goals and rules must initiate the project. A rubric would be helpful in providing students with the structure to assess their own progress as a group.

- Provide problem-solving tasks within a meaningful, realistic context in order to facilitate transfer of learning. Students need to view math as a necessary skill in their lives. Assist transference through the use of problem-solving tasks such as finding the best mortgage deal or comparing the cost of different types of transportation.

- Develop students' skills in interpreting numerical or graphical information appearing within documents and text. Math does not always take the form of computation. Graphs, tables, text, payment schedules, and contracts are just a few of the ways in which text is filled with mathematical concepts. Strategies to use in teaching students how to accurately interpret such documents can include having students graph information from their lives for the last 24 hours. Pictorial, circle, line, or any type of graph can be used to visually document numerical information. Another activity would be to have students critique and discuss an article filled with numerical information such as an employee benefit statement.

Math Journals in the GED® Classroom

Writing activities can help students better understand the material they are trying to learn and ultimately can shift students from looking at math as a series of formulas that have to be solved or computations that must be completed to recognizing that mathematics is a process. Most GED® students do not recognize that mathematics is a process; rather, they see each problem with a specific answer and no real relationship among the wide range of problems that they encounter in the classroom, on tests, or in the real world.

Math journals can be used for many purposes. Look at math journals as variables rather than constants, providing opportunities for students to:

- Increase their feelings of confidence in being able to learn and use mathematical concepts and skills to solve a wide range of problems and thus help alleviate math anxiety.
- Be more aware of what they do and do not know.
- Make use of their own prior knowledge when solving new problems.
- Identify their own questions about an area with which they are less familiar.
- Develop their ability to think through a problem and identify possible methods for solving it.
- Collect and organize their thoughts.
- Monitor their own progress as they gain higher-level problem-solving skills and are able to work with more complex problems.
- Make connections between mathematical ideas as they write about various strategies that could be used for problem solving.
- Communicate more precisely how they think.

In *Writing in the Mathematics Curriculum* (Burchfield, Jorgensen, McDowell, and Rahn 1993), the authors identify three possible categories for math journal prompts. These categories include:

- Affective/attitudinal prompts, which focus on how students feel.
- Mathematical content prompts, which focus on what the material is about.
- Process prompts, which require students to explain what they are thinking and doing.

Using Affective/Attitudinal Prompts in Math Journals

Many adult learners are math phobic or, at least, fearful of trying and failing to solve problems. Their own feelings of inability to learn mathematics get in their way and, in essence, become a self-fulfilling prophecy. The more anxious the learner becomes, the less he/she is able to focus on the math content. Affective/attitudinal math journal prompts enable students to express their feelings, concerns, and fears about mathematics.

The following are a few examples of affective/attitudinal prompts:

- Explain how you feel when you begin a math session.
- One secret I have about math is . . .
- If I become better at math, I can . . .
- My best experience with math was when . . .
- My worst experience with math was when . . .
- Describe how it feels if you have to show your work on the board . . .
- One math activity that I really enjoyed was . . .

Using Mathematical Content Prompts in Math Journals

When working with math content, most adult learners expect merely to perform a series of computations and provide a specific answer. Rarely have they been asked to explain what they did to find an answer. Mathematical content prompts provide learners with an opportunity to explain how they arrived at a specific answer, thus enabling them to begin making connections between what they have done and the math content itself. These types of prompts also enable students to support their point of view or to explain errors they made in their calculations.

Mathematical content prompts can be as simple as students writing definitions in their own terms, such as defining geometric shapes or providing math examples of what variables are and why they are used.

The following are a few examples of mathematical content prompts:

- The difference between . . . and . . . is . . .
- How do you . . .?
- What patterns did you find in…?
- How do you use … in everyday life?
- Explain in your own words what . . . means.
- One thing I have to remember with this kind of problem is . . .
- Why do you have to . . .?

Using Process Prompts in Math Journals

Process prompts allow learners to explore how they go about solving a problem. It moves them from mere computations to looking at math problem solving as a process that, just as in solving real-life problems, requires a series of steps and questions that must be analyzed and answered.

Process prompts require learners to look more closely at how they think.

The following are examples of process prompts:

- How did you reach the answer for the problem about . . . ?
- What part in solving the problem was the easiest? What was the most difficult? Why?
- The most important part of solving this problem was . . .
- Provide instructions for a fellow student to use to solve a similar problem.
- What would happen if you missed a step in the problem? Why?
- What decisions did you have to make to solve this type of problem?
- When I see a word problem, the first thing I do is . . .
- Review what you did today and explain how it is similar to something you already knew.
- Is there a shortcut for finding . . .? What is it? How does it work?
- Could you find the answer to this problem another way?
- I draw pictures or tables to solve problems because . . .
- To solve today's math starter, I had to . . .
- The first answer I found for this problem was not reasonable, so I had to . . . [18]

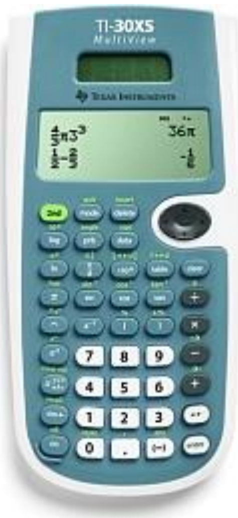
[18] Math Journals for All Ages. http://math.about.com/aa123001a.htm

Burchfield, P.C., Jorgensen, P.R., McDowell, K.G., and Rahn, J. (1993). *Writing in the Mathematics Curriculum*

Countryman, J. (1992). *Writing to Learn Mathematics.* Portsmouth, NH: Heinemann

Whitin, Phyllis and Whitin, David J. (2000). *Math Is Language Too: Talking and Writing in the Mathematics Classroom.* Urbana, IL: National Council of Teachers of English, and Reston, VA: National Council of Teachers of Mathematics

The Tools of Math – Using the TI-30XS MultiView™ Calculator



The tools of math are everywhere. We use calculators, spreadsheets, databases, software, and notebooks. Think about how these tools are used in daily life, in the workplace, and in academic settings.

It's important to remember that the GED® test is designed to reflect what graduating seniors know and can do. The use of the scientific calculator is common in most high school level math courses. The 2014 GED® Mathematics Test allows the use of the TI-30XS MultiView™ calculator on all but the first five test questions. Students are provided with directions on how to use the basic functions of the calculator. However, it is important that students be familiar with this important tool.

Although candidates will use a virtual calculator on the test, practicing with the handheld version of the calculator will allow them to become familiar with the basic functions.

Resources

For additional information on the TI-30XS MultiView™ calculator, access the following sites for free tutorials, videos, teacher guides, and lesson plans:

- GED Testing Service® – Videos on the TI-30XS MultiView™ calculator
  http://www.gedtestingservice.com/testers/calculator

- Texas Instruments – Teacher Guide for the TI-30XS MultiView™ calculator
  http://education.ti.com/en/us/guidebook/details/en/62522EB25D284112819FDB8A46F90740/30x_mv_tg

- Texas Instruments – Free lesson plans for the TI-30XS MultiView™ calculator
  http://education.ti.com/en/us/activity/search/subject?d=2717022CF3A841848A4DEF66BB848B88&s=B843CE852FC5447C8DD88F6D1020EC61

- Atomic Learning® TI-30XS MultiView™ calculator
  http://www.atomiclearning.com/k12/en/ti30xs

Using Manipulatives in the Classroom

A mathematical manipulative is defined as any material or object from the real world that students move around to show a mathematics concept. Research indicates that students of all ages can benefit by first being introduced to mathematical concepts through physical exploration. By planning lessons that proceed from concrete to pictorial to abstract representations of concepts, you can make content mastery more accessible to students of all ages.

With concrete exploration (through touching, seeing, and doing), students can gain deeper and longer-lasting understandings of math concepts. For example, students can explore the concept of least common multiple with integer bars. They can place the integer bars side by side, experiment, and discover how to create a combination of bars that are the same length. Once students have a concrete understanding of the concept of greatest common factor (GCF) as matching lengths, they will find it easier to use a number line or make lists to identify the GCF. Similarly, if students use grid paper, pencils, and scissors to discover the formulas for computing the areas of parallelograms, triangles, or trapezoids, the formulas will make sense to them and they will be more likely to remember the formulas.

Using manipulative materials in teaching mathematics will help students learn:

- To relate real-world situations to mathematics symbolism
- To work together cooperatively in solving problems
- To discuss mathematical ideas and concepts
- To verbalize their mathematics thinking
- To make presentations in front of a large group
- That there are many different ways to solve problems
- That mathematics problems can be symbolized in many different ways
- That they can solve mathematics problems without just following teachers' directions

Managing Manipulatives

Using manipulatives can present classroom management challenges. Teachers find that manipulatives can get lost or broken. Students sometimes use manipulatives for other than the intended purpose. Distributing the manipulatives can take time, but the following guidelines can assist the teacher in using them more effectively.

- *Set Up Simple Storage Systems* - Set up a simple system to store the manipulatives. Some teachers arrange shelves or cupboards with plastic boxes or snap-and-seal bags. Others place their materials in the center of tables or desks. Clearly label your storage containers. Make sure students understand the system and have easy access to it.
- *Establish Clear Rules* - Prior to your first use of manipulatives, discuss a clear set of rules for using the manipulatives with your students. You may want to explain what manipulatives will be used for and include the following information:
    - appropriate uses for learning

- handling
- storage
- distribution and return
- student roles and responsibilities

Structured Learning Experiences

The key to successful hands-on activities is to provide a structured learning experience in which students learn how to use manipulatives. To maximize learning, always provide three levels of practice.

- Modeled Tasks - Before distributing materials, provide clear instructions and model the tasks the students will carry out. If it is appropriate, you can invite students to help you model. For example, if the students are going to use fraction bars to complete addition problems, you might have students model using overhead bars and how they completed the process.
- Guided Practice - Give students opportunities to practice prior to working individually or in small groups. If this is the first time the student is handling the manipulative, consider allowing extra time for exploration. You might ask the students to construct the largest possible right angle on a geoboard and give them time to figure out how to work with the pegs and rubber bands. Monitor student practice during this phase to give them the support they need to be successful when they work independently.
- Independent Work - Once students know how to use manipulatives, they can complete problems on their own or in small groups with less support. This is an excellent time to informally assess learning and provide intervention as needed.

Resources

For additional information on manipulatives, access:

- National Library of Virtual Manipulatives for Math
  http://nlvm.usu.edu/en/nav/index.html

- Algebra 4 All
  http://a4a.learnport.org/page/algebra-tiles

- Working with Algebra Tiles
  http://mathbits.com/MathBits/AlgebraTiles/AlgebraTiles.htm

Evidence-Based Practices for the Math Classroom

The following is an overview of evidence-based practices for effective math instruction.

Instructional Element	Recommended Practices
Curriculum Design	Ensure mathematics curriculum is based on challenging content Ensure curriculum is standards based Clearly identify skills, concepts and knowledge to be mastered Ensure that the mathematics curriculum is vertically and horizontally articulated
Professional Development for Teachers	Provide professional development which focuses on: Knowing/understanding standards Using standards as a basis for instructional planning Teaching using best practices Multiple approaches to assessment Develop/provide instructional support materials such as curriculum maps and pacing guides and provide math coaches
Technology	Provide professional development on the use of instructional technology tools Provide student access to a variety of technology tools Integrate the use of technology across all mathematics curricula and courses
Manipulatives	Use manipulatives to develop understanding of mathematical concepts Use manipulatives to demonstrate word problems Ensure use of manipulatives is aligned with underlying math concepts
Instructional Strategies	Focus lessons on specific concept/skills that are standards based Differentiate instruction through flexible grouping, individualizing lessons, compacting, using tiered assignments, and varying question levels Ensure that instructional activities are learner-centered and emphasize inquiry/problem-solving Use experience and prior knowledge as a basis for building new knowledge Use cooperative learning strategies and make real life connections

	Use scaffolding to make connections to concepts, procedures and understanding Ask probing questions which require students to justify their responses Emphasize the development of basic computational skills
Assessment	Ensure assessment strategies are aligned with standards/concepts being taught Evaluate both student progress/performance and teacher effectiveness Utilize student self-monitoring techniques Provide guided practice with feedback Conduct error analyses of student work Utilize both traditional and alternative assessment strategies Ensure the inclusion of diagnostic, formative and summative strategies Increase use of open-ended assessment techniques

Geometric Thinking Skills

Geometry is the attempt to understand space, shape, and dimension. It's about the properties of objects (their angles and surfaces, for instance) and the consequences of how these objects are positioned (where their shadows fall, how people must move around them).

Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world. Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Students who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas, as well as other advanced mathematical topics.

The Development of Geometric Thinking

Geometry curriculum is often presented through the memorization and application of formulas, axioms, theorems, and proofs. This type of instruction requires that students function at a formal deductive level. However, many students lack the prerequisite skills and understanding of geometry in order to operate at this level.

The work of two Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, has had a major impact on the design of geometry instruction and curriculum. The van Hiele's work began in 1959 and has since become the most influential factor in the American geometry curriculum.

The van Hiele model is a five-level hierarchy of ways of understanding spatial ideas. Each of the five levels describes the thinking processes used in geometric contexts. The levels describe how one thinks and what types of geometric ideas one thinks about, rather than how much knowledge one has. Remember that the levels are not age dependent, but are rather sequential in nature.

Level 1: Visualization

The objects of thought at level 1 are shapes and what they "look like."

Students recognize and name figures based on the global, visual characteristics of the figure—a gestalt-like approach to shape. Students operating at this level are able to make measurements and even talk about properties of shapes, but these properties are not thought about explicitly. It is the appearance of the shape that defines it for the student. A square is a square "because it looks like a square." Because appearance is dominant at this level, appearances can overpower properties of a shape. For example, a square that has been rotated so that all sides are at a 45° angle to the vertical may not appear to be a square for a level 1 thinker. Students at this level will sort and classify shapes based on their appearances—"I put these together because they all look sort of alike."

The products of thought at level 1 are classes or groupings of shapes that seem to be "alike."

Level 2: Analysis

The objects of thought at level 2 are classes of shapes rather than individual shapes.

Students at the analysis level are able to consider all shapes within a class rather than a single shape. Instead of talking about *this* rectangle, it is possible to talk about *all* rectangles. By focusing on a class of shapes, students are able to think about what makes a rectangle a rectangle (four sides, opposite sides parallel, opposite sides same length, four right angles, congruent diagonals, etc.). The irrelevant features (e.g., size or orientation) fade into the background. At this level, students begin to appreciate that a collection of shapes goes together because of properties. Ideas about an individual shape can now be generalized to all shapes that fit that class. If a shape belongs to a particular class such as cubes, it has the corresponding properties of that class. "All cubes have six congruent faces, and each of those faces is a square." These properties were only implicit at level 0. Students operating at level 2 may be able to list all the properties of squares, rectangles, and parallelograms but not see that these are subclasses of one another that all squares are rectangles and all rectangles are parallelograms. In defining a shape, level 2 thinkers are likely to list as many properties of a shape as they know.

The products of thought at level 2 are the properties of shapes.

Level 3: Informal Deduction

The objects of thought at level 3 are the properties of shapes.

As students begin to be able to think about properties of geometric objects without the constraints of a particular object, they are able to develop relationships between and among these properties. "If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, it must be a rectangle." With greater ability to engage in "if-then" reasoning, shapes can be classified using only minimum characteristics. For example, four congruent sides and at least one right angle can be sufficient to define a square. Rectangles are parallelograms with a right angle. Observations go beyond properties themselves and begin to focus on logical arguments *about* the properties. Students at level 3 will be able to follow and appreciate an informal deductive argument about shapes and their properties. "Proofs" may be more intuitive than rigorously deductive. However, there is an appreciation that a logical argument is compelling. An appreciation of the axiomatic structure of a formal deductive system, however, remains under the surface.

The products of thought at level 3 are relationships among properties of geometric objects.

Level 4: Deduction

The objects of thought at level 4 are relationships among properties of geometric objects.

At level 4, students are able to examine more than just the properties of shapes. Their earlier thinking has produced conjectures concerning relationships among properties. Are these conjectures correct? Are they "true"? As this analysis of the informal arguments takes place, the structure of a system complete with axioms, definitions, theorems, corollaries, and postulates begins to develop and can be appreciated as the necessary means of establishing geometric truth. At this level, students begin to appreciate the need for a system of logic that rests on a minimum set of assumptions and from which other truths can be derived. The student at this level is able to work with abstract statements about geometric properties and make conclusions based more on logic than intuition. This is the level of the traditional high school geometry course. A student operating at level 4 can clearly observe that the diagonals of a rectangle bisect each other, just as a student at a lower level of thought can. However, at level 4, there is an appreciation of the need to prove this from a series of deductive arguments. The level 3 thinker, by contrast, follows the argument but fails to appreciate the need.

The products of thought at level 4 are deductive axiomatic systems for geometry.

Level 5: Rigor

The objects of thought at level 5 are deductive axiomatic systems for geometry.

At the highest level of the van Hiele hierarchy, the object of attention is axiomatic systems themselves, not just the deductions within a system. There is an appreciation of the distinctions and relationships between different axiomatic systems. This is generally the level of a college mathematics major who is studying geometry as a branch of mathematical science.

The products of thought at level 5 are comparisons and contrasts among different axiomatic systems of geometry.

Teaching Geometric Thinking: A Few Strategies

Research has shown that the use of the following strategies is effective in assisting students to learn concepts, discover efficient procedures, reason mathematically, and become better problem solvers in the areas of geometry and measurement.

Have high expectations for all students. Ensure that students' learning styles are addressed in teaching geometry. By varying instructional strategies and presenting content in a range of formats, teachers can better meet the needs and address the learning styles of individual students. Incorporate academic standards so that lessons can be selected that are necessary for the student to learn.

Base practice on educational research. Incorporating research results and findings is a way to profit from the work of others. Research indicates that students benefit from cooperative learning types of activities with the opportunity to connect those activities to real-world situations.

Integrate content areas. The learning of geometric ideas becomes more meaningful for students when it is presented in contexts beyond individual lessons. Mathematics should be connected in three ways:

- within mathematical concepts;
- with other disciplines; and
- to real-world situations.

All of these, in different ways, help students to establish a framework of strategies that students can call upon in order to solve new problems and learn new concepts and algorithms. Integrating mathematics into real-world situations also assists students to better "connect" to what is being taught and to answer the question of "Why do I need to learn this?" It is also important to incorporate the best possible materials into the geometry curriculum, drawing from resources available as well as real-world materials.

Encourage cooperative learning and collaboration with others. Research supports that students learn best when they have time to explore and discover concepts. Cooperative learning is a valuable tool for learning, as students learn both from the teacher and from each other. Cooperative learning also actively involves students in the learning process and encourages them to communicate mathematically. A teacher who promotes mathematical reasoning and problem solving also tends to create a classroom that is a supportive and collegial community of learners.

When a teacher poses challenging problems to a class, students benefit from working in small groups to explore and discuss ideas and then reporting their findings to the class. It is also effective to put students into groups, in which they compare and contrast the ways they approached problems and arrived at a solution. These strategies support the sharing of diverse kinds of thinking, place value on listening to and learning from others, and help students to develop ways to solve future problems. Cooperative learning also prepares students to work as a team, which is something that many employers will expect from them later, as employees.

Use technology as a tool. Technology provides a unique opportunity to improve student performance in mathematical reasoning and problem solving. In geometry, interactive software can enhance student understanding of such things as multi-dimensional shapes and their properties. Virtual manipulatives allow students to better understand specific topics in the area of geometry. The Internet also provides an excellent tool for students and teachers to use to access information and to communicate with others.

Use inquiry-based learning. Teaching is most engaging for students when their own thoughts, opinions, and curiosities are addressed in the subject at hand. The best way to ensure that students feel they have a stake in their own learning is to create a classroom that values exploration, where teacher and students alike can support, discuss, and evaluate ways of thinking in an open and ongoing way.

The difference between traditional and inquiry-based learning in a geometry classroom would be as follows:

In a traditional classroom, students first learn about discrete concepts and procedures, such as the perimeter and area of a rectangle. They would then learn how to use the formula "A = L x W" to find the area of a rectangle, given its length and width. Later, students would learn about the area of a triangle and how to find the area using a formula. Eventually, students would apply this knowledge to determine the area of a figure composed of a rectangle and triangle.

When a teacher uses inquiry-based learning, the process is reversed. The teacher presents a problem first, such as, "If you want to paint the front of a house, how much area must you paint?" Students then explore the problem and – and with the teacher's guidance – discover that they need to understand how to cover an area with a standard-size unit. After solving the problem, students look for efficient procedures for finding the area and then develop formulas accordingly.

Promote mathematical reasoning and problem-solving skills. Critical thinking is a crucial component of learning. Students should be encouraged to justify their thinking, rather than merely providing a correct answer to a problem. Through encouraging mathematical reasoning and problem solving, students will increase their ability to solve problems, which in turn builds confidence. A teacher should:

Use problems from a variety of sources to introduce new geometric concepts

- Pose questions frequently
- Encourage students to think for themselves
- Present problems that are open-ended (whenever feasible), to allow for multiple problem-solving approaches.

Use hand-on activities to model topics. Use hands-on activities to model concepts in geometry and measurement and to help students better understand the concepts of mathematics. Students grow to understand concepts when they have first experienced concepts on a concrete level. Students' long-term use of concrete instructional materials and manipulatives supports achievement in mathematics.

Cluster concepts. When students learn concepts and relationships in isolation, they often forget these ideas or are slow in making the connections among them. A thinking process called "clustering," used to group, unify, integrate, and/or make connections among concepts, is something that students and adults use routinely, often without even realizing it. A teacher can use this approach to cluster mathematical ideas, concepts, relationships, and objects in order to reveal common characteristics. This presentation in turn helps students to categorize and classify those ideas or objects and to remember properties and attributes, which makes the learning more meaningful. An example would be to teach quadrilaterals as a unifying concept where students compare and contrast all quadrilaterals according to their attributes so that students can internalize the idea and develop a hierarchy based on the figures' properties.

Reflect on learning. Metacognitive strategies increase students' learning. Have students reflect on and communicate what they have learned and what is still unclear. To assist students in "thinking about their thinking," have them:

- Make connections between new information and known ideas
- Choose appropriate thinking strategies for a particular use
- Plan, monitor, and assess how effective certain thinking processes were

One way of accomplishing this kind of reflection is through the use of student-created portfolios. Even the process of selection that goes into making portfolios helps the student to build self-awareness and ultimately gives the student more control over her or his own learning. Writing encourages students to analyze, communicate, discover, and organize their growing knowledge.

Integrate assessment and instruction. Ongoing classroom assessment promotes the learning process. Combine traditional modes of assessment with geometry assignments that require open-ended answers and constructed responses. The latter encourage students to:

- Incorporate higher-order thinking and skills into their solutions
- Communicate their geometric thinking
- Explore various strategies to a solution
- Apply their existing knowledge
- Organize, analyze, and interpret information
- Create a mathematical model
- Make and test predictions

Use of Manipulatives and Real-World Scenarios

Teachers are always interested in looking for ways to improve their teaching and to help students understand mathematics. Research in England, Japan, China, and the United States supports the idea that mathematics instruction and student mathematics understanding will be more effective if manipulative materials are used.

A mathematical manipulative is defined as any material or object from the real world that students move around to show a mathematics concept.

Research indicates that students of all ages can benefit by first being introduced to mathematical concepts through physical exploration. By planning lessons that proceed from concrete to pictorial to abstract representations of concepts, you can make content mastery more accessible to students of all ages.

Long-Lasting Understandings

With concrete exploration (through touching, seeing, and doing), students can gain deeper and longer-lasting understandings of math concepts. For example, if students use grid paper, pencils, and scissors to discover the formulas for computing the areas of parallelograms, triangles, or trapezoids, the formulas will make sense to them and they will be more likely to remember the formulas.

A Sample Activity

Have students "build" the ideal school. Building uses many of the geometric formulas and applications with which students need to be familiar. Begin by having students think about what it takes to build a school. Have them develop questions that must be asked, such as:

- What shape will the building be?
- How many classrooms are needed?
- How many square feet are required for the appropriate number of classrooms?
- How many and what size of windows and doors will be used?
- What size parking lot is needed?

Have students develop a blueprint that includes their ideas for the "perfect school."

Algebraic Thinking

Algebra is fundamental to understanding mathematical thought. Algebraic thinking includes:

- "looking for structure (patterns and regularities) to make sense of situations
- generalizing beyond the specific by using symbols for variable quantities
- representing relationships systematically with tables, graphs, and equations
- reasoning logically to address/solve new problems"[19]

Most people recognize that algebra is needed by scientists or engineers, but algebraic thinking and reasoning is also used by health care providers, home builders, graphic designers, and in daily life. Algebraic thinking is necessary – not just for a few professions – but for today's workforce. Blanton and Kaput (2003) stated that teachers must find ways to support algebraic thinking and create a classroom culture that values "students modeling, exploring, arguing, predicting, conjecturing, and testing their ideas, as well as practicing computational skills."

How does one get started "algebrafying" the adult education classroom? Think about transforming current activities and word problems from a single answer to opportunities to discover patterns and make generalizations about mathematical facts and relationships. Use prompts that extend student thinking, such as:

- Tell me what you were thinking.
- Could you solve this in a different way?
- How do you know that is true?
- Does that always work?

This type of relational thinking is necessary for students in the GED® preparatory classroom, as well as for adults in the workplace.

The Big Ideas

The key prerequisites for students to be successful in the study of algebra are to understand the big ideas of algebra – variable, symbolic notation, and multiple representations. When teaching algebra, use practical experiences that go beyond the mere computation required by equations.

When developing practice activities in the algebra classroom, be sure to:

- Develop processes/procedures for students to use when approaching algebraic tasks
- Create authentic exercises that highlight the critical attributes related to the concept being taught
- Provide opportunities for students to verbalize the task and predict what type of answer is expected
- Offer opportunities for students to discuss and write responses to questions dealing with key concepts being learned
- Select authentic exercises that anticipate future skills to be learned

[19] National Institute for Literacy. Algebraic Thinking in Adult Education. (2010). Washington, DC

- Design authentic exercises that integrate a number of ideas to reinforce prior learning as well as current and future concepts

Create exercises that highlight the critical attributes related to the skill or concept being taught:

- Provide opportunities for students to verbalize about the task and predict what type of answer is expected
- Offer opportunities for students to discuss and write responses to questions dealing with key concepts being learned
- Select exercises that anticipate future skills to be learned
- Design exercises that integrate a number of ideas to reinforce prior learning as well as current, and future concepts

As students learn algebraic concepts, they need to develop different procedures to use. Being able to recognize a pattern is an important critical thinking skill in solving certain algebraic problems.

- **Finding** patterns involves looking for regular features of a situation that repeats.
- **Describing** patterns involves communicating the regularity in words or in a mathematically concise way that other people can understand.
- **Explaining** patterns involves thinking about why the pattern continues forever, even if one has not exhaustively looked at each one.
- **Predicting** with patterns involves using your description to predict pieces of the situation that are not given.

A Sample Activity

Provide students with real-world problems that can be solved using algebraic thinking skills, such as the following problem.

My Ford Bronco was fitted at the factory with 30 inch diameter tires. That means its speedometer is calibrated for 30 inch diameter tires. I "enhanced" the vehicle with All Terrain tires that have a 31 inch diameter. How will this change the speedometer readings? Specifically, assuming the speedometer was accurate in the first place, what should I make the speedometer read as I drive with my 31 inch tires so that the actual speed is 55 mph?

CTL Resources for Algebra. The Department of Mathematics. Education University of Georgia
http://jwilson.coe.uga.edu/ctl/ctl/resources/Algebra/Algebra.html

Resources

Although there are numerous resources to assist the classroom instructor, the following are a few websites to start with:

- Annenberg Learner
  http://www.learner.org/index.html

- Illuminations
  http://illuminations.nctm.org

- Inside Mathematics
  http://insidemathematics.org/index.php/home

- Khan Academy
  http://www.khanacademy.org/

- The Math Dude
  http://www.montgomeryschoolsmd.org/departments/itv/MathDude/MD_Downloads.shtm

- Math Planet
  http://library.thinkquest.org/16284/index_s.htm

- Geometry Center
  http://www.geom.uiuc.edu/

- Online Resources for Teaching and Strengthening Fundamental, Quantitative, Mathematical, and Statistical Skills. NICHE
  http://serc.carleton.edu/NICHE/teaching_materials_qr.html#partone

- Key Elements to Algebra Success
  http://ntnmath.keasmath.com/

- Algebra Nation
  http://www.algebranation.com/